Lifetime asymptotics of iterated Brownian motion in $\mathbb{R}^n$

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Abstract

Let $\tau_D(Z)$ be the first exit time of iterated Brownian motion from a domain $D \subset \mathbb{R}^n$ started at $z \in D$ and let $P_z[\tau_D(Z) > t]$ be its distribution. In this paper we establish the exact asymptotics of $P_z[\tau_D(Z) > t]$ over bounded domains as an improvement of the results in [12, 24], for $z \in D$

$$\lim_{t\to\infty} t^{-1/2} \exp(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3})P_z[\tau_D(Z) > t] = C(z),$$

where $C(z) = (\lambda_D 2^{7/2})/\sqrt{3\pi}\left(\psi(z)\int_D \psi(y)dy\right)^2$. Here λ_D is the first eigenvalue of the Dirichlet Laplacian $\frac{1}{2}\Delta$ in D, and ψ is the eigenfunction corresponding to λ_D .

We also study lifetime asymptotics of Brownian-time Brownian motion (BTBM), $Z_t^1 = z + X(|Y(t)|)$, where X_t and Y_t are independent one-dimensional Brownian motions.

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1 Introduction and statement of main results

Iterated Brownian motion (IBM) have attracted the interest of several authors [1, 2, 3, 7, 8, 9, 10, 12, 15, 18, 23, 24, 27, 29]. Several other iterated processes including Brownian-time Brownian motion (BTBM) have also been studied [1, 2, 19, 25, 26]. One of the main differences between these iterated processes and Brownian motion is that they are not Markov processes. However, these processes have many properties similar to that of Brownian motion (see [2, 3, 12, 23], and references therein).

To define iterated Brownian motion Z_t started at $z \in \mathbb{R}$, let X_t^+, X_t^- and Y_t be three independent one-dimensional Brownian motions, all started at 0. Two-sided Brownian motion is defined by

$$X_t = \begin{cases} X_t^+, & t \geq 0 \\ X_{(-t)}^-, & t < 0. \end{cases}$$

Then iterated Brownian motion started at $z \in \mathbb{R}$ is

$$Z_t = z + X(Y_t), \quad t \geq 0.$$

In $\mathbb{R}^n$, one requires $X^\pm$ to be independent $n-$dimensional Brownian motions. This is the version of the iterated Brownian motion due to Burdzy, see [7].

We next define another closely related process, the so called Brownian-time Brownian motion. Let X_t and Y_t be two independent one-dimensional Brownian motions, all started at 0. Brownian-time Brownian motion started at $z \in \mathbb{R}$ is

$$Z_t^1 = z + X(|Y_t|) \quad t \geq 0.$$

In $\mathbb{R}^n$ one requires X to be an $n-$dimensional Brownian motion.

Let τ_D be the first exit time of Brownian motion from a domain $D \subset \mathbb{R}^n$. The large time behavior of $P_z[\tau_D > t]$ has been studied for several types of domains, including general cones [5, 11], parabola-shaped domains [4, 22], twisted domains [13], unbounded convex domains [21] and bounded domains [28]. Our aim in this article is to do the same for the first exit time of IBM over bounded domains in $\mathbb{R}^n$, and for the first exit time of BTBM over several domains in $\mathbb{R}^n$. See Bañuelos and DeBlassie [3], Li [21], Lifshits and Shi [22] and Nane [23] for a survey of results obtained for Brownian motion and iterated Brownian motion in these domains.

For many bounded domains $D \subset \mathbb{R}^n$ the asymptotics of $P_z[\tau_D > t]$ is well-known (See [28] for a more precise statement of this.) For $z \in D$,

$$\lim_{t\to\infty} e^{\lambda_D t} P_z[\tau_D > t] = \psi(z) \int_D \psi(y)dy, \tag{1.1}$$

where λ_D is the first eigenvalue of $\frac{1}{2}\Delta$ with Dirichlet boundary conditions and ψ is its corresponding eigenfunction.

DeBlassie [12] proved that for iterated Brownian motion in bounded domains; for $z \in D$,

$$\lim_{t\to\infty} t^{-1/3} \log P_z[\tau_D(Z) > t] = -\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}. \tag{1.2}$$

The limits (1.1) and (1.2) are very different in that the latter involves taking the logarithm which may kill many unwanted terms in the exponential. It is then natural to ask if it is possible to obtain an analogue of (1.1) for IBM. That is, to remove the log in (1.2). In [24], we improved the limit in (1.2) as follows; for $z \in D$,

$$\begin{aligned} 2C(z) &\le \liminf_{t\to\infty} t^{-1/2} \exp(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}) P_z[\tau_D(Z) > t] \\ &\le \limsup_{t\to\infty} t^{-1/2} \exp(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}) P_z[\tau_D(Z) > t] \le \pi C(z), \end{aligned}$$

where $C(z) = \lambda_D \sqrt{2\pi/3}\left(\psi(z)\int_D \psi(y)dy\right)^2$.

In this paper we prove the following theorem which improves both limits above.

Theorem 1.1. *Let $D \subset \mathbb{R}^n$ be a domain for which (1.1) holds pointwise and let λ_D and ψ be as above. Then for $z \in D$,*

$$\lim_{t\to\infty} t^{-1/2} \exp(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}) P_z[\tau_D(Z) > t] = \frac{(\lambda_D 2^{7/2})}{\sqrt{3\pi}} \left(\psi(z)\int_D \psi(y)dy\right)^2.$$

Remark 1.1. *Observe that $2\lambda_D\sqrt{2\pi/3} \le (\lambda_D 2^{7/2})/\sqrt{3\pi} \le \pi\lambda_D\sqrt{2\pi/3}$, so Theorem 1.1 is in agreement with the results obtained previously in [12, 24].*

In [13], DeBlassie and Smits studied the tail behavior of the first exit time of Brownian motion in twisted domains in the plane. Let $D \subset \mathbb{R}^2$ be a

domain whose boundary consists of three curves (in polar coordinates)

$$\begin{array}{lll} C_1: & \theta = f_1(r), & r \geq r_1 \\ C_2: & \theta = f_2(r), & r \geq r_1 \\ C_3: & r = r_1, & f_2(r) \leq \theta \leq f_1(r) \end{array}$$

where f_1 and f_2 are smooth and the curves C_1 and C_2 do not cross:

$$0 < f_1(r) - f_2(r) < \pi, \quad r \geq r_1.$$

DeBlassie and Smits call D a twisted domain if there are constants $r_0 > 0$, $\gamma > 0$ and $p \in (0, 1]$ and a smooth function $f(r)$ such that the curves $f_1(r)$ and $f_2(r)$, $r \geq r_0$, are obtained from $f(r)$ by moving out $\pm\gamma r^p$ units along the normal to the curve $\theta = f(r)$ at the point whose polar coordinates are $(r, f(r))$. They call γr^p the growth radius and $\theta = f(r)$ the generating curve. DeBlassie and Smits [13, Theorem 1.1] have the following tail behavior of the first exit time of Brownian motion in twisted domains $D \subset \mathbb{R}^2$ with growth radius γr^p, $\gamma > 0$, $0 < p < 1$

$$\lim_{t\to\infty} t^{-(\frac{1-p}{1+p})} \log P_z[\tau_D > t] = -l_1 = -\left[\frac{\pi^{2p-1}}{\gamma 2^{2p}(1-p)^{2p}}\right]^{\frac{2}{p+1}} C_p \tag{1.3}$$

where

$$C_p = (1+p)\left[\frac{\pi^{2+p}}{8^p p^{2p}(1-p)^{1-p}} \frac{\Gamma^{2p}\left(\frac{1-p}{2p}\right)}{\Gamma^{2p}\left(\frac{1}{2p}\right)}\right]^{\frac{1}{p+1}}.$$

For these domains, Nane [23] obtained the following; for all $z \in D$,

$$\lim_{t\to\infty} t^{-(\frac{1-p}{3+p})} \log P_z[\tau_D(Z) > t] = -(\frac{3+p}{2+2p})(\frac{1+p}{1-p})^{(\frac{1-p}{3+p})} \pi^{(\frac{2-2p}{3+p})} l_1^{(\frac{2+2p}{3+p})},$$

where l_1 is the limit given by (1.3).

We obtained in [24], the following for BTBM in twisted domains; for $z \in D$,

$$\lim_{t\to\infty} t^{-(\frac{1-p}{p+3})} \log P_z[\tau_D(Z^1) > t] = -2^{(\frac{2p-2}{3+p})}(\frac{3+p}{2+2p})(\frac{1+p}{1-p})^{(\frac{1-p}{3+p})} \pi^{(\frac{2-2p}{3+p})} l_1^{(\frac{2+2p}{3+p})},$$

where l_1 is the limit given by the limit given by (1.3).

DeBlassie and Smits [13] also obtained similar results for $p=1$. Let $D\subset\mathbb{R}^2$ be a twisted domain with growth radius γr, $\gamma>0$. Then for $z\in D$,

$$\lim_{t\to\infty}[\log t]^{-1}\log P_z[\tau_D>t]=-C(\gamma)=-\pi\left[4\arccos\frac{1}{\sqrt{1+\gamma^2}}\right]^{-1}. \qquad (1.4)$$

We obtain the following lifetime asymptotics of BTBM in twisted domains.

Theorem 1.2. *Let $D\subset\mathbb{R}^2$ be a twisted domain with growth radius γr, $\gamma>0$. Then for $z\in D$,*

$$\lim_{t\to\infty}[\log t]^{-1}\log P_z[\tau_D(Z^1)>t]=-C(\gamma)/2,$$

where $C(\gamma)$ as in (1.4).

Using Theorem 1.3. from [24], which says that for all $z\in D$ and all $t>0$,

$$P_z[\tau_D(Z)>t]\ \le\ 2P_z[\tau_D(Z^1)>t],$$

we obtain the following for IBM in twisted domains.

Corollary 1.1. *Let $D\subset\mathbb{R}^2$ be a twisted domain with growth radius γr, $\gamma>0$. Then for $z\in D$,*

$$\limsup_{t\to\infty}[\log t]^{-1}\log P_z[\tau_D(Z)>t]\le -C(\gamma)/2,$$

where $C(\gamma)$ as in (1.4).

In [21], using Gaussian techniques, Li studied lifetime asymptotics of Brownian motion in domains of the following form

$$P_f=\{(x,y)\in\mathbb{R}^{n+1}:y>f(x),x\in\mathbb{R}^n\}$$

for $f(x)=\exp(|x|^p), p>0$. Li established that for $z\in P_f$,

$$\lim_{t\to\infty}t^{-1}(\log t)^{2/p}\log P_z[\tau_{P_f}>t]=-j_\nu^2, \qquad (1.5)$$

where $\nu=(n-2)/2$ and j_ν is the smallest positive zero of the Bessel function J_v.

We obtain the following theorem in these domains

Theorem 1.3. *Let P_f be as above with $f(x) = \exp(|x|^p), p > 0$. Then for $z \in P_f$,*

$$\lim_{t\to\infty} t^{-1/3}(\log t)^{4/3p} \log P_z[\tau_{P_f}(Z^1) > t] = -C(p),$$

where

$$C(p) = (3/2)^{(4+3p)/3p} 2^{1/3} (j_\nu^2 2^{2/p})^{2/3} (\pi^2/8)^{1/3}.$$

Using Theorem 1.3. from [24], we obtain the following for IBM in these domains.

Corollary 1.2. *Let P_f be as above with $f(x) = \exp(|x|^p), p > 0$. Then for $z \in P_f$,*

$$\limsup_{t\to\infty} t^{-1/3}(\log t)^{4/3p} \log P_z[\tau_{P_f}(Z) > t] \le -C(p).$$

For $f(x) = h(|x|)$ and $h^{-1}(x) = Ax^\alpha(\log x)^\beta$, $x > 1$. Li obtained the following: let $\epsilon > 0$. For t large, $z \in P_f$

$$-(1+\epsilon)C_{\alpha,\beta,1} \le t^{-\frac{(1-\alpha)}{(1+\alpha)}} (\log t)^{\frac{2\beta}{(1+\alpha)}} \log P_z[\tau_{P_f} > t] \le -(1-\epsilon)C_{\alpha,\beta,2},. \quad (1.6)$$

where

$$C_{\alpha,\beta,1} = 2^{-1}(1-\alpha)^{-1}(\alpha^{-\alpha}(1+\alpha)^{2\beta+2}A^{-2}j_\nu^2)1/(1+\alpha)$$

and

$$C_{\alpha,\beta,2} = (1+\alpha)(2\alpha)^{-\alpha/(1+\alpha)}(2^{-1}(1+\alpha))^{2\beta/(1+\alpha)})C^{1/(1+\alpha)}$$

where $C = (1-\alpha)^{-1}2^{2\beta-1}A^{-2}j_\nu^2$.

We have the following for BTBM in these domains.

Theorem 1.4. *For $0 < \alpha < 1$ and $\beta \in \mathbb{R}$,*

$$\begin{aligned} -C(1) &\le \liminf_{t\to\infty} t^{-(1-\alpha)/(3+\alpha)} (\log t)^{4\beta(1+\alpha)/(3+\alpha)} \log P_z[\tau_{P_f}(Z^1) > t] \\ &\le \limsup_{t\to\infty} t^{-(1-\alpha)/(3\alpha+1)} (\log t)^{4\beta(1+\alpha)/(3+\alpha)} \log P_z[\tau_{P_f}(Z^1) > t] \\ &\le -C(2) \end{aligned}$$

where

$$C(1) = \left(\frac{3+\alpha}{2(1+\alpha)}\right)^{\frac{3+\alpha+4\beta}{3+\alpha}} \left(\frac{1-\alpha}{(3+\alpha)}\right) (\pi^2/8)^{\frac{1-\alpha}{(3+\alpha)}} (C_{\alpha,\beta,1})^{\frac{2(1+\alpha)}{(3+\alpha)}} 2^{\frac{4\beta}{(3+\alpha)}},$$

and

$$C(2) = \left(\frac{3+\alpha}{2(1+\alpha)}\right)^{\frac{3+\alpha+4\beta}{3+\alpha}} \left(\frac{1-\alpha}{(3+\alpha)}\right) (\pi^2/8)^{\frac{1-\alpha}{(3+\alpha)}} (C_{\alpha,\beta,2})^{\frac{2(1+\alpha)}{(3+\alpha)}} 2^{\frac{4\beta}{(3+\alpha)}}.$$

Using Theorem 1.3. from [24], we obtain the following for IBM in these domains.

Corollary 1.3. *For $0 < \alpha < 1$ and $\beta \in \mathbb{R}$. Let P_f be as above with $f(x) = h(|x|)$, $h^{-1}(x) = Ax^\alpha (\log x)^\beta$, $x > 1$. Then for $z \in P_f$,*

$$\limsup_{t\to\infty} t^{-(1-\alpha)/(3\alpha+1)} (\log t)^{4\beta(1+\alpha)/(3+\alpha)} \log P_z[\tau_{P_f}(Z^1) > t] \leq -C(2)$$

where $C(2)$ is as above.

The paper is organized as follows. In §2, we give some preliminary lemmas to be used in the proof of main results. Theorem 1.1 is proved in §3. §4 is devoted to prove Theorems 1.2, 1.3 and 1.4. In §5, we recall several asymptotic results to be used in the proof of main results from Nane [24].

2 Preliminaries

In this section we state some preliminary facts that will be used in the proof of main results.

In what follows we will write $f \approx g$ and $f \lesssim g$ to mean that for some positive C_1 and C_2, $C_1 \leq f/g \leq C_2$ and $f \leq C_1 g$, respectively. We will also write $f(t) \sim g(t)$, as $t \to \infty$, to mean that $f(t)/g(t) \to 1$, as $t \to \infty$.

The main fact is the following Tauberian theorem ([14, Laplace transform method, 1958, Chapter 4]). Laporte [20] also studied this type of integrals. Let h and f be continuous functions on $\mathbb{R}$. Suppose f is non-positive and has a global max at x_0, $f'(x_0) = 0$, $f''(x_0) < 0$ and $h(x_0) \neq 0$ and $\int_{-\infty}^{\infty} h(x)\exp(\lambda f(x)) < \infty$ for all $\lambda > 0$. Then as $\lambda \to \infty$,

$$\int_0^\infty h(x)\exp(\lambda f(x))dx \sim h(x_0)\exp(\lambda f(x_0))\sqrt{\frac{2\pi}{\lambda|f''(x_0)|}}. \tag{2.1}$$

It can be easily seen from Laplace transform method that as $\lambda \to \infty$,

$$\int_0^\infty \exp(-\lambda(x + x^{-2}))dx \sim \exp(-3\lambda 2^{-2/3})\sqrt{\frac{2^{4/3}\pi}{3\lambda}}. \tag{2.2}$$

Similarly, as $t \to \infty$,

$$\int_0^\infty \exp(-\frac{at}{u^2} - bu)du \ \sim \sqrt{\frac{\pi}{3}} 2^{2/3} a^{1/6} b^{-2/3} t^{1/6} \exp(-3a^{1/3} b^{2/3} 2^{-2/3} t^{1/3}). \tag{2.3}$$

This follows from equation (2.2) and after making the change of variables $u = (atb^{-1})^{1/3}x$.

Finally, we obtain, as $t \to \infty$,

$$\int_0^\infty u \exp(-\frac{at}{u^2} - bu)du \ \sim 2\sqrt{\frac{\pi}{3}} a^{1/2} b^{-1} t^{1/2} \exp(-3a^{1/3}b^{2/3}2^{-2/3}t^{1/3}). \quad (2.4)$$

Writing the power series for the cosine function we easily see that as $t \to \infty$,

$$\int_0^\infty x \cos(\pi K/x) \exp(-\frac{\pi^2 t}{2x^2} - \lambda_D x)dx$$
$$\sim 2\sqrt{\frac{\pi}{3}} (\frac{\pi^2}{2})^{1/2} \lambda_D^{-1} t^{1/2} \exp(-\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}). \quad (2.5)$$

Next we give a Lemma that will be used in the proof of Theorem 1.2.

Lemma 2.1. *Let ξ be a positive random variable such that as $x \to 0^+$*

$$[|\log x|]^{-1} \log P[\xi \le x] \sim -c/2.$$

Then as $\lambda \to \infty$

$$[\log \lambda]^{-1} \log E[\exp(-\lambda\xi)] \sim -c/2.$$

Proof. Let $\epsilon > 0$, then by hypothesis there exists $\delta(\epsilon) > 0$ such that

$$\delta^{c(1+\epsilon)/2} \le P[\xi \le \delta] \le \delta^{c(1-\epsilon)/2},$$

for all $\delta < \delta(\epsilon)$. Let f be the density of ξ. Then

$$\begin{aligned} E[\exp(-\lambda\xi)] &= \int_0^\delta e^{-\lambda x} f(x)dx + \int_\delta^\infty e^{-\lambda x} f(x)dx \\ &\le P[\xi \le \delta] + \exp(-\delta\lambda) \end{aligned}$$

Now we use the fact that $\exp(-x) \le c_N x^{-N}$ for any $N \in \mathbb{N}$ and for some $c_N > 0$.

Hence

$$E[\exp(-\lambda\xi)] \ \le \ \delta^{c(1-\epsilon)/2} + c_N(\delta\lambda)^{-N}.$$

To minimize this upper bound we require

$$\delta^{c(1-\epsilon)/2} = c_N(\delta\lambda)^{-N}$$

which gives

$$\delta = (c_N(\lambda)^{-N})^{1/(N+c(1-\epsilon)/2)}.$$

Hence for some $D(N) > 0$,

$$E[\exp(-\lambda\xi)] \leq D(N)\lambda^{-\frac{Nc(1-\epsilon)/2}{(N+c(1-\epsilon)/2)}}$$

Taking logarithm of both sides, dividing by $\log\lambda$, and letting $\lambda \to \infty$, we obtain

$$\limsup_{t\to\infty} [\log\lambda]^{-1} \log E[\exp(-\lambda\xi)] \leq -\frac{Nc(1-\epsilon)/2}{(N+c(1-\epsilon)/2)}$$

Now letting $N \to \infty$, and $\epsilon \to 0$, we get

$$\limsup_{t\to\infty} [\log\lambda]^{-1} \log E[\exp(-\lambda\xi)] \leq -c/2.$$

Lower bound follows from

$$\begin{aligned} E[\exp(-\lambda\xi)] &\geq \int_0^\delta e^{-\lambda x} f(x)dx \\ &\geq e^{-\delta\lambda} P[\xi \leq \delta] \geq \delta^{c(1+\epsilon)/2} e^{-\delta\lambda} \end{aligned}$$

and taking $\delta = \lambda^{-1}$, for large λ, we get

$$E[\exp(-\lambda\xi)] \gtrsim \lambda^{-c(1+\epsilon)/2}.$$

Taking logarithm of both sides and dividing by $\log\lambda$, and letting $\epsilon \to 0$, we get

$$\liminf_{t\to\infty} [\log\lambda]^{-1} E[\exp(-\lambda\xi)] \geq -c/2.$$

$\square$

We next state a version of de Bruijn's Tauberian Theorem (Kasahara [17, Theorem 3] and Bingham, Goldie and Teugels [6, p. 254].)

Theorem 2.1 (de Bruijn Tauberian Theorem). *Let ξ be a positive random variable. Then, for $\alpha > 0$ and $\beta \in \mathbb{R}$*

$$\log P[\xi \leq \epsilon] \sim -C\epsilon^{-\alpha}|\log \epsilon|^{\beta} \quad as \quad \epsilon \to 0^{+}$$

if and only if

$$\log E[\exp(-\lambda\xi)] \sim -(1+\alpha)^{1-\beta/(1+\alpha)}\alpha^{-\alpha/(1+\alpha)}C^{1/(1+\alpha)}\lambda^{\alpha/(1+\alpha)}(\log \lambda)^{\beta/(1+\alpha)}$$

as $\lambda \to \infty$.

We give next an application of de Bruijn's Tauberian Theorem.

Lemma 2.2. *Let ξ be a positive random variable with density $f(x) = \gamma e^{-v}V$, $v(x) = Cx^{-1/2}(\log x^{-1/2})^{-2/p}$, and $dv = -V dx$ Then as $x \to 0^{+}$*

$$P[\xi \leq x] \sim -Cx^{1/2}(|\log x^{1/2}|)^{-2/p}.$$

In this case

$$\log E[\exp(-\lambda\xi)] \sim -(3/2)^{(4+3p)/3p}2^{1/3}(C2^{2/p})^{2/3}\lambda^{1/3}(\log \lambda)^{-4/3p}$$

3 Iterated Brownian motion in bounded domains

If $D \subset \mathbb{R}^n$ is an open set, write

$$\tau_D^{\pm}(z) = \inf\{t \geq 0 : \quad X_t^{\pm} + z \notin D\},$$

and if $I \subset \mathbb{R}$ is an open interval, write

$$\eta_I = \eta(I) = \inf\{t \geq 0 : \quad Y_t \notin I\}.$$

Recall that $\tau_D(Z)$ stands for the first exit time of iterated Brownian motion from D. As in DeBlassie [12, §3.], we have by the continuity of the paths for $Z_t = z + X(Y_t)$, if f is the probability density of $\tau_D^{\pm}(z)$

$$P_z[\tau_D(Z) > t] = \int_0^{\infty}\int_0^{\infty} P_0[\eta_{(-u,v)} > t]f(u)f(v)dvdu. \tag{3.1}$$

The proof of Theorem 1.1 . The following is well-known

$$P_0[\eta_{(-u,v)} > t] = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\exp(-\frac{(2n+1)^2\pi^2}{2(u+v)^2}t)\sin\frac{(2n+1)\pi u}{u+v}, \quad (3.2)$$

(see Feller [16, pp. 340-342]).
Let $\epsilon > 0$. From Lemma 5.1, choose $M > 0$ so large that

$$(1-\epsilon)\frac{4}{\pi}e^{-\frac{\pi^2 t}{2}}\sin\pi x \le P_x[\eta_{(0,1)} > t] \le (1+\epsilon)\frac{4}{\pi}e^{-\frac{\pi^2 t}{2}}\sin\pi x, \quad (3.3)$$

for $t \ge M$, uniformly $x \in (0,1)$. For a bounded domain with regular boundary it is well-known (see [28, page 121-127]) that there exists an increasing sequence of eigenvalues, $\lambda_1 < \lambda_2 \le \lambda_3 \cdots$, and eigenfunctions ψ_k corresponding to λ_k such that,

$$P_z[\tau_D \le t] = \sum_{k=1}^{\infty}\exp(-\lambda_k t)\psi_k(z)\int_D \psi_k(y)dy. \quad (3.4)$$

From the arguments in DeBlassie [12, Lemma A.4]

$$f(t) = \frac{d}{dt}P_z[\tau_D \le t] = \sum_{k=1}^{\infty}\lambda_k\exp(-\lambda_k t)\psi_k(z)\int_D \psi_k(y)dy. \quad (3.5)$$

Finally choose $K > 0$ so large that

$$A(z)(1-\epsilon)\exp(-\lambda_D u) \le f(u) \le A(z)(1+\epsilon)\exp(-\lambda_D u)$$

for all $u \ge K$, where

$$A(z) = \lambda_1\psi_1(z)\int_D \psi_1(y)dy = \lambda_D\psi(z)\int_D \psi(y)dy.$$

We further assume that t is so large that $K < \frac{1}{2}\sqrt{t/M}$. Define A for $K > 0$ and $M > 0$ as

$$A = \left\{(u,v): \quad K \le u \le \frac{1}{2}\sqrt{\frac{t}{M}},\ u \le v \le \sqrt{\frac{t}{M}} - u\right\}.$$

By equation (3.3) and from equation (3.10) in [12],

$$P_z[\tau_D(Z) > t] = 2\int_0^\infty \int_u^\infty P_{\frac{u}{u+v}}[\eta_{(0,1)} > \frac{t}{(u+v^2)}]f(u)f(v)dvdu$$

$$\geq \ C^1 \int_K^{\frac{1}{2}\sqrt{t/M}} \int_u^{\sqrt{t/M}-u} \sin\left(\frac{\pi u}{(u+v)}\right)\exp(-\frac{\pi^2 t}{2(u+v)^2})\exp(-\lambda_D(u+v))dvdu,$$

where $C^1 = C^1(z) = 2(4/\pi)A(z)^2(1-\epsilon)^3$. Changing the variables $x = u+v, z = u$ the integral is

$$= C^1 \int_K^{\frac{1}{2}\sqrt{t/M}} \int_{2z}^{\sqrt{t/M}} \sin\left(\frac{\pi z}{x}\right)\exp(-\frac{\pi^2 t}{2x^2})\exp(-\lambda_D x)dxdz,$$

and reversing the order of integration

$$= C^1 \int_{2K}^{\sqrt{t/M}} \int_K^{\frac{1}{2}x} \sin\left(\frac{\pi z}{x}\right)\exp(-\frac{\pi^2 t}{2x^2})\exp(-\lambda_D x)dzdx$$

$$= C^1/\pi \int_{2K}^{\sqrt{t/M}} x\cos\left(\frac{\pi K}{x}\right)\exp(-\frac{\pi^2 t}{2x^2})\exp(-\lambda_D x)dx$$

From equation (2.5) as $t \to \infty$,

$$\int_0^\infty x\cos\left(\frac{\pi K}{x}\right)\exp(-\frac{\pi^2 t}{2x^2})\exp(-\lambda_D x)dx$$
$$\sim \ 2\sqrt{\frac{\pi}{3}}(\frac{\pi^2}{2})^{1/2}\lambda_D^{-1}t^{1/2}\exp(-\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}). \tag{3.6}$$

Now for some $c_1 > 0$,

$$\int_0^{K/\delta} x\exp(-\frac{\pi^2 t}{2x^2} - \lambda_D x)dx$$
$$\leq e^{-\pi^2\delta^2 t/2K^2}\int_0^{K/\delta} x\exp(-\lambda_D x)dx \lesssim e^{-c_1 t}, \tag{3.7}$$

and

$$\int_{\sqrt{t/M}}^\infty x\exp(-\frac{\pi^2 t}{2x^2})\exp(-\lambda_D x)dx \leq \int_{\sqrt{t/M}}^\infty x\exp(-\lambda_D x)dx$$
$$= (\sqrt{t/M}\lambda_D^{-1} + \lambda_D^{-2})\exp(-\lambda_D\sqrt{t/M}). \tag{3.8}$$

Now from equations (3.6)-(3.8) we get

$$\liminf_{t\to\infty} t^{-1/2}\exp(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3})P_z[\tau_D(Z)>t]\geq (C^1/\pi)2\sqrt{\frac{\pi}{3}}(\frac{\pi^2}{2})^{1/2}\lambda_D^{-1}. \tag{3.9}$$

For the upper bound for $P[\tau_D(Z)>t]$ from equation (3.10) in [12],

$$P_z[\tau_D(Z)>t]=2\int_0^\infty\int_u^\infty P_{\frac{u}{u+v}}[\eta_{(0,1)}>\frac{t}{(u+v^2)}]f(u)f(v)dvdu\ . \tag{3.10}$$

We define the following sets that make up the domain of integration,

$$\begin{aligned}
A_1 &= \{(u,v): v\geq u\geq 0,\ u+v\geq\sqrt{t/M}\},\\
A_2 &= \{(u,v):\ u\geq 0,\ v\geq K,\ u\leq v,\ u+v\leq\sqrt{t/M}\},\\
A_3 &= \{(u,v):\ 0\leq u\leq v\leq K\}.
\end{aligned}$$

Over the set A_1 we have for some $c>0$,

$$\begin{aligned}
&\int\int_{A_1} P_{\frac{u}{u+v}}[\eta_{(0,1)}>\frac{t}{(u+v)^2}]f(u)f(v)dvdu\\
&\leq\int\int_{A_1} f(u)f(v)dvdu\leq\exp(-c\sqrt{t/M}).
\end{aligned}\tag{3.11}$$

The equation (3.11) follows from the distribution of τ_D from Lemma 2.1 in [23].

Since on A_3, $t/(u+v)^2\geq M$,

$$\begin{aligned}
&\int\int_{A_3} P_{\frac{u}{u+v}}[\eta_{(0,1)}>\frac{t}{(u+v)^2}]f(u)f(v)dvdu\\
&\leq\int_0^K\int_0^K\exp(-\frac{\pi^2 t}{2(u+v)^2})f(u)f(v)dvdu.\\
&\leq\exp(-\frac{\pi^2 t}{8K^2})\int_0^K\int_0^K f(u)f(v)dvdu\leq\exp(-\frac{\pi^2 t}{8K^2}).
\end{aligned}\tag{3.12}$$

Let $C_1 = C_1(z) = 2(4/\pi)A(z)^2(1+\epsilon)^3$. For the integral over A_2 we get,

$$\begin{aligned}
&\int\int_{A_2} P_{\frac{u}{u+v}}[\eta_{(0,1)} > \frac{t}{(u+v)^2}]f(u)f(v)dvdu \\
\leq\ & C_1 \int_0^K \int_K^{\sqrt{t/M}-u} f(u)\exp(-\frac{\pi^2 t}{2(u+v)^2} - \lambda_D v)dvdu \\
+\ & C_1 \int_K^{1/2\sqrt{t/M}} \int_u^{\sqrt{t/M}-u} \sin\left(\frac{\pi u}{u+v}\right)\exp(-\frac{\pi^2 t}{2(u+v)^2} - \lambda_D(u+v))dvdu \\
=\ & I + II. \qquad (3.13)
\end{aligned}$$

Changing variables $u+v=z$, $u=w$

$$\begin{aligned}
I &= \int_0^K \int_K^{\sqrt{t/M}-u} \exp(-\frac{\pi^2 t}{2(u+v)^2})f(u)\exp(-\lambda_D v)dvdu \\
&\leq \int_0^K \int_{w+K}^{\sqrt{t/M}} \exp(-\frac{\pi^2 t}{2z^2})f(w)\exp(-\lambda_D z)\exp(\lambda_D w)dzdw \\
&\leq \exp(\lambda_D K)\int_0^K f(w)dw \int_0^\infty \exp(-\frac{\pi^2 t}{2z^2})\exp(-\lambda_D z)dz \\
&\lesssim t^{1/6}\exp(-\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}). \qquad (3.14)
\end{aligned}$$

Equation (3.14) follows from equation (2.3), with $a = \pi^2/2$, $b = \lambda_D$.

Changing variables $u+v=z$, $u=w$

$$\begin{aligned}
II &\leq C_1 \int_K^{1/2\sqrt{t/M}} \int_{2w}^{\sqrt{t/M}} \sin\left(\frac{\pi w}{z}\right)\exp(-\frac{\pi^2 t}{2z^2} - \lambda_D z)dzdw \\
&= C_1 \int_{2K}^{\sqrt{t/M}} \int_K^{z/2} \sin\left(\frac{\pi w}{z}\right)\exp(-\frac{\pi^2 t}{2z^2} - \lambda_D z)dwdz \qquad (3.15) \\
&\leq C_1/\pi \int_{2K}^{\sqrt{t/M}} z\cos\left(\frac{\pi K}{z}\right)\exp(-\frac{\pi^2 t}{2z^2} - \lambda_D z)dz \\
&\leq (1+\epsilon)(C_1/\pi)2\sqrt{\frac{\pi}{3}}(\frac{\pi^2}{2})^{1/2}\lambda_D^{-1}t^{1/2}\exp(-\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3}). \qquad (3.16)
\end{aligned}$$

Equation (3.15) follows by changing the order of the integration. Equation (3.16) follows from equation (2.5).

Now from equations (3.11), (3.12), (3.14) and (3.16) we obtain

$$\limsup_{t\to\infty} t^{-1/2}\exp(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3})P_z[\tau_D(Z)>t] \le (1+\epsilon)(\frac{C_1}{\pi})2\sqrt{\frac{\pi}{3}}(\frac{\pi^2}{2})^{1/2}\lambda_D^{-1}. \tag{3.17}$$

Finally, from equations (3.9) and (3.17) and letting $\epsilon \to 0$,

$$\begin{aligned} C(z) &\le \liminf_{t\to\infty} t^{-1/2}\exp(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3})P_z[\tau_D(Z)>t] \\ &\le \limsup_{t\to\infty} t^{-1/2}\exp(\frac{3}{2}\pi^{2/3}\lambda_D^{2/3}t^{1/3})P_z[\tau_D(Z)>t] \le C(z), \end{aligned}$$

where $C(z) = (\lambda_D 2^{7/2})/\sqrt{3\pi}\left(\psi(z)\int_D \psi(y)dy\right)^2$. □

4 Brownian-time Brownian motion in unbounded domains

In this section we study Brownian-time Brownian motion (BTBM), Z_t^1 started at $z \in \mathbb{R}$, in several unbounded domains.

If $D \subset \mathbb{R}^n$ is an open set, write

$$\tau_D(z) = \inf\{t \ge 0 : \ X_t + z \notin D\},$$

and if $I \subset \mathbb{R}$ is an open interval, we write

$$\eta_I = \inf\{t \ge 0 : \ Y_t \notin I\}.$$

Let $\tau_D(Z^1)$ stand for the first exit time of BTBM from D. We have by the continuity of paths

$$P_z[\tau_D(Z^1)>t] = P[\eta(-\tau_D(z),\tau_D(z))>t]. \tag{4.1}$$

Proof of Theorem 1.2. Let $\epsilon > 0$. From Lemma 5.2, choose $M > 0$ so large that

$$(1-\epsilon)\exp(-\frac{\pi^2 t}{8u^2})\frac{\pi t}{u^3} \le \frac{d}{du}P_0[\eta_{(-u,u)}>t] \le (1+\epsilon)\ \exp(-\frac{\pi^2 t}{8u^2})\frac{\pi t}{u^3}$$

for all $u \le \sqrt{t/M}$.

Let $C = C(\gamma)$. From the hypothesis choose $K > 0$ so large that

$$u^{-C(1+\epsilon)} \le P(\tau_D(z) > u) \le u^{-C(1-\epsilon)} \quad \text{for} \quad u \ge K. \tag{4.2}$$

We further assume that t is so large that $K < \sqrt{t/M}$.

$$\begin{aligned} P_z[\tau_D(Z^1) > t] &= \int_0^\infty (\frac{d}{du} P_0[\eta_{(-u,u)} > t]) P[\tau_D(z) > u] du \\ &\gtrsim t \int_K^{\sqrt{t/M}} \exp(-\frac{\pi^2 t}{8u^2}) u^{-(C(1+\epsilon)+3)} du \end{aligned} \tag{4.3}$$

changing variables $u^{-2} = x$, $du = -1/2 x^{-3/2} dx$ the integral is

$$\gtrsim t \int_{M/t}^{K^{-2}} \exp(-\frac{\pi^2 t x}{8}) x^{C(1+\epsilon)/2} dx \tag{4.4}$$

Changing variables, $z = \pi^2 t x/8$, the integral is

$$\gtrsim t^{-C(1+\epsilon)/2} \int_{\pi^2 M/8}^{K^{-2}\pi^2 t/8} e^{-z} z^{C(1+\epsilon)/2} dz. \tag{4.5}$$

Now since for some $c_0 > 0$,

$$\int_{K^{-2}\pi^2 t/8}^\infty e^{-z} z^{C(1+\epsilon)/2} dz \le e^{-c_0 t},$$

$$\int_0^{\pi^2 M/8} e^{-z} z^{C(1+\epsilon)/2} dz < \infty,$$

and

$$\int_0^\infty e^{-z} z^{C(1+\epsilon)/2} dz = \Gamma(1 + C(1+\epsilon)/2).$$

We have

$$P_z[\tau_D(Z^1) > t] \gtrsim t^{-C(1+\epsilon)/2}. \tag{4.6}$$

We now give an upper bound.

$$\begin{aligned} P_z[\tau_D(Z^1) > t] &= \int_0^\infty P_0(\eta_{(-u,u)} > t) f(u) du \\ &\lesssim \int_0^{\sqrt{t/M}} e^{-\frac{\pi^2 t}{8u^2}} f(u) du + \int_{\sqrt{t/M}}^\infty f(u) du \\ &\lesssim E\left[\exp\left(-\frac{\pi^2 t}{8(\tau_D(z))^2}\right)\right] + (\sqrt{t/M})^{-C(1-\epsilon)} \\ &\lesssim t^{-C(1-\epsilon)/2} \end{aligned} \tag{4.7}$$

Equation (4.7) follows from Lemma 2.1 and the asymptotics of $\tau_D(z)$.
Now from Equations (4.6) and (4.7) we have

$$t^{-C(1+\epsilon)/2} \lesssim P_z[\tau_D(Z^1) > t] \ \lesssim \ t^{-C(1-\epsilon)/2}.$$

Now taking logarithm of the above inequalities and then dividing by $\log t$ and letting $\epsilon \to 0$, we obtain the desired result. □

Proof of Theorem 1.3. Let $\epsilon > 0$. From Lemma 5.2, choose $M > 0$ so large that

$$(1-\epsilon)\exp(-\frac{\pi^2 t}{8u^2})\frac{\pi t}{u^3} \leq \frac{d}{du}P_0[\eta_{(-u,u)} > t] \leq (1+\epsilon)\ \exp(-\frac{\pi^2 t}{8u^2})\frac{\pi t}{u^3} \tag{4.8}$$

for all $u \leq \sqrt{t/M}$.

Let $C = j_\nu^2$. From the hypothesis choose $K > 0$ so large that

$$\begin{aligned} &\exp(-C(1+\epsilon)u(\log u)^{-2/p}) \\ &\leq P(\tau_{P_f}(z) > u) \\ &\leq \exp(-C(1-\epsilon)u(\log u)^{-2/p}) \end{aligned} \tag{4.9}$$

for $u \geq K$. We further assume that t is so large that $K < \sqrt{t/M}$.

Then, by equations (4.8) and (4.9)

$P_z[\tau_D(Z^1) > t]$

$$\gtrsim \ t\int_K^{\sqrt{t/M}} u^{-3}\exp(-\frac{\pi^2 t}{8u^2})\exp(-C(1+\epsilon)u(\log u)^{-2/p})du \tag{4.10}$$

changing variables $u^{-2} = x$, $du = -1/2x^{-3/2}dx$ the integral is

$$\gtrsim \ t\int_{M/t}^{K^{-2}} \exp(-\frac{\pi^2 tx}{8})\exp(-C(1+\epsilon)x^{-1/2}(\log x^{-1/2})^{-2/p})dx$$

Now we set

$$v(x) = C(1+\epsilon)x^{-1/2}(\log x^{-1/2})^{-2/p}$$

which gives

$$dv = C(1+\epsilon)x^{-3/2}(\log x^{-1/2})^{-2/p}[-1/2 + 1/p(\log x^{-1/2})^{-1}]dx = -V dx$$

Then the integral is

$$\gtrsim \quad t \int_{M/t}^{K^{-2}} VV^{-1} \exp(-\frac{\pi^2 tx}{8}) \exp(-C(1+\epsilon)x^{-1/2}(\log x^{-1/2})^{-2/p})dx$$

Now

$$V^{-1} \gtrsim t^{-3/2}[1/2 - 1/p(\log\sqrt{t/M})^{-1}]^{-1} \gtrsim t^{-1/2}$$

Hence the integral is

$$\gtrsim \quad t^{1/2} \int_{M/t}^{K^{-2}} \exp(-\frac{\pi^2 tx}{8}) \exp(-v) V dx \tag{4.11}$$

Now from Lemma 2.2 and de Bruijn's Tauberian Theorem 2.1 we have

$$\log \int_0^\infty \exp(-\frac{\pi^2 tx}{8}) \exp(-v)(-dv) \sim -C(p,\epsilon)t^{1/3}(\log t)^{-4/(3p)}$$

where $C(p,\epsilon) = (3/2)^{(4+3p)/3p} 2^{1/3}((1+\epsilon)j_\nu^2 2^{2/p})^{2/3}(\pi^2/8)^{1/3}$.

From the bounds for some $c_1, c_2 > 0$,

$$\int_0^{M/t} \exp(-\frac{\pi^2 tx}{8}) \exp(-v)(-dv) \leq e^{-c_1 t^{1/2}(\log t)^{-2/p}},$$

and

$$\int_{K^{-2}}^\infty \exp(-\frac{\pi^2 tx}{8}) \exp(-v)(-dv) \leq e^{-c_2 t},$$

we get

$$P_z[\tau_{P_f}(Z^1) > t] \gtrsim \exp(-(1+\epsilon)^2 C(p,\epsilon)t^{1/3}(\log t)^{-4/3p}) \tag{4.12}$$

We next give the upper bound

$$\begin{aligned}
P_z[\tau_{P_f}(Z^1) > t] &= \int_0^\infty P_0(\eta_{(-u,u)} > t) f(u) du \\
&\lesssim \int_0^{\sqrt{t/M}} e^{-\frac{\pi^2 t}{8u^2}} f(u) du + \int_{\sqrt{t/M}}^\infty f(u) du \\
&\lesssim E\left[\exp\left(-\frac{\pi^2 t}{8(\tau_{P_f}(z))^2}\right)\right] + \exp(-C(1-\epsilon)t^{1/2}(\log t)^{-2/p})
\end{aligned}$$

The upper bound follows from de Bruijn's Tauberian Theorem by observing that as $x \to 0^+$

$$\log P[1/(\tau_{P_f}(z))^2 \leq x] = \log P[\tau_{P_f}(z) \geq x^{-1/2}] \sim -Cx^{-1/2}(|\log x^{-1/2}|)^{-2/p}$$

Hence

$$P_z[\tau_{P_f}(Z^1) > t] \lesssim \exp(-(1-\epsilon)D(p,\epsilon)t^{1/3}(\log t)^{-4/3p}) \tag{4.13}$$

where $D(p,\epsilon) = (3/2)^{(4+3p)/3p}2^{1/3}((1-\epsilon)j_\nu^2 2^{2/p})^{2/3}(\pi^2/8)^{1/3}$.

Therefore from equations (4.12) and (4.13), we obtain

$$\begin{aligned}
&\exp(-(1+\epsilon)^2 C(p,\epsilon)t^{1/3}(\log t)^{-4/3p}) \\
&\lesssim P_z[\tau_{P_f}(Z^1) > t] \\
&\lesssim \exp(-(1-\epsilon)^2 D(p,\epsilon)t^{1/3}(\log t)^{-4/3p})
\end{aligned}$$

Now taking logarithms and letting $\epsilon \to 0$, we get the desired result. $\square$

Proof of Theorem 1.4. The proof follows the same steps of the proof of Theorem 1.3, so we omit the details. $\square$

5 Asymptotics

In this Section we will recall some lemmas that were used in section 3 and section 4. The following lemma is proved in [12, Lemma A1] (it also follows from more general results on "intrinsic ultracontractivity"). We include it for completeness.

Lemma 5.1. *As $t \to \infty$,*

$$P_x[\eta_{(0,1)} > t] \;\sim\; \frac{4}{\pi}e^{-\frac{\pi^2 t}{2}}\sin \pi x, \quad \text{uniformly for} \quad x \in (0,1).$$

We recall a result from Nane [23, Lemma 6.2] that will be used for the process Z^1.

Lemma 5.2. *Let $B = \{u > 0 : \; t/u^2 > M\}$ for M large. Then on B,*

$$\frac{d}{du}P_0[\eta_{(-u,u)} > t] \sim \exp(-\frac{\pi^2 t}{8u^2})\frac{\pi t}{u^3}. \tag{5.1}$$

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